

# Package ‘rotasym’

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**License** GPL-3

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rotasym-package	<b>rotasym</b> – <i>Tests for Rotational Symmetry on the Hypersphere</i>
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### Description

Implementation of the tests for rotational symmetry on the hypersphere proposed in García-Portugués, Paindaveine and Verdebout (2020) <doi:10.1080/01621459.2019.1665527>. The package implements the proposed distributions on the hypersphere based on the tangent-normal decomposition. It also allows for the replication of the data application considered in the paper.

### Author(s)

Eduardo García-Portugués, Davy Paindaveine, and Thomas Verdebout.

### References

García-Portugués, E., Paindaveine, D., Verdebout, T. (2020) On optimal tests for rotational symmetry against new classes of hyperspherical distributions. *Journal of the American Statistical Association*, 115(532):1873–1887. doi: [10.1080/01621459.2019.1665527](https://doi.org/10.1080/01621459.2019.1665527)

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ACG	<i>Angular central Gaussian distribution</i>
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### Description

Density and simulation of the Angular Central Gaussian (ACG) distribution on  $S^{p-1} := \{\mathbf{x} \in R^p : \|\mathbf{x}\| = 1\}$ ,  $p \geq 1$ . The density at  $\mathbf{x} \in S^{p-1}$ ,  $p \geq 2$ , is given by

$$c_{p,\Lambda}^{\text{ACG}} (\mathbf{x}' \Lambda^{-1} \mathbf{x})^{-p/2} \quad \text{with} \quad c_{p,\Lambda}^{\text{ACG}} := 1/(\omega_p |\Lambda|^{1/2})$$

where  $\Lambda$  is the shape matrix, a  $p \times p$  symmetric and positive definite matrix, and  $\omega_p$  is the surface area of  $S^{p-1}$ .

**Usage**

`d_ACG(x, Lambda, log = FALSE)`

`c_ACG(p, Lambda, log = FALSE)`

`r_ACG(n, Lambda)`

**Arguments**

<code>x</code>	locations in $S^{p-1}$ to evaluate the density. Either a matrix of size $c(n_x, p)$ or a vector of length $p$ . Normalized internally if required (with a warning message).
<code>Lambda</code>	the shape matrix $\Lambda$ of the ACG. A symmetric and positive definite matrix of size $c(p, p)$ .
<code>log</code>	flag to indicate if the logarithm of the density (or the normalizing constant) is to be computed.
<code>p</code>	dimension of the ambient space $R^p$ that contains $S^{p-1}$ . A positive integer.
<code>n</code>	sample size, a positive integer.

**Details**

Due to the projection of the ACG, the shape matrix  $\Lambda$  is only identified up to a constant, that is,  $\Lambda$  and  $c\Lambda$  give the same ACG distribution. Usually,  $\Lambda$  is normalized to have trace equal to  $p$ .

`c_ACG` is vectorized on  $p$ . If  $p = 1$ , then the ACG is the uniform distribution in the set  $\{-1, 1\}$ .

**Value**

Depending on the function:

- `d_ACG`: a vector of length  $n_x$  or 1 with the evaluated density at  $x$ .
- `r_ACG`: a matrix of size  $c(n, p)$  with the random sample.
- `c_ACG`: the normalizing constant.

**Author(s)**

Eduardo García-Portugués, Davy Paindaveine, and Thomas Verdebout.

**References**

Tyler, D. E. (1987). Statistical analysis for the angular central Gaussian distribution on the sphere. *Biometrika*, 74(3):579–589. doi: [10.1093/biomet/74.3.579](https://doi.org/10.1093/biomet/74.3.579)

**See Also**

[tangent-elliptical](#).

**Examples**

```

# Simulation and density evaluation for p = 2
Lambda <- diag(c(5, 1))
n <- 1e3
x <- r_ACG(n = n, Lambda = Lambda)
col <- viridisLite::viridis(n)
r <- runif(n, 0.95, 1.05) # Radius perturbation to improve visualization
plot(r * x, pch = 16, col = col[rank(d_ACG(x = x, Lambda = Lambda))])

# Simulation and density evaluation for p = 3
Lambda <- rbind(c(5, 1, 0.5),
               c(1, 2, 1),
               c(0.5, 1, 1))
x <- r_ACG(n = n, Lambda = Lambda)
if (requireNamespace("rgl")) {
  rgl::plot3d(x, col = col[rank(d_ACG(x = x, Lambda = Lambda))], size = 5)
}

```

cosines-signs

*Cosines and multivariate signs of a hyperspherical sample about a given location*

**Description**

Computation of the cosines and multivariate signs of the hyperspherical sample  $\mathbf{X}_1, \dots, \mathbf{X}_n \in S^{p-1}$  about a location  $\boldsymbol{\theta} \in S^{p-1}$ , for  $S^{p-1} := \{\mathbf{x} \in R^p : \|\mathbf{x}\| = 1\}$  with  $p \geq 2$ . The *cosines* are defined as

$$V_i := \mathbf{X}_i' \boldsymbol{\theta}, \quad i = 1, \dots, n,$$

whereas the *multivariate signs* are the vectors  $\mathbf{U}_1, \dots, \mathbf{U}_n \in S^{p-2}$  defined as

$$\mathbf{U}_i := \boldsymbol{\Gamma}_{\boldsymbol{\theta}} \mathbf{X}_i / \|\boldsymbol{\Gamma}_{\boldsymbol{\theta}} \mathbf{X}_i\|, \quad i = 1, \dots, n.$$

The projection matrix  $\boldsymbol{\Gamma}_{\boldsymbol{\theta}}$  is a  $p \times (p-1)$  semi-orthogonal matrix that satisfies

$$\boldsymbol{\Gamma}_{\boldsymbol{\theta}}' \boldsymbol{\Gamma}_{\boldsymbol{\theta}} = \mathbf{I}_{p-1} \quad \text{and} \quad \boldsymbol{\Gamma}_{\boldsymbol{\theta}} \boldsymbol{\Gamma}_{\boldsymbol{\theta}}' = \mathbf{I}_p - \boldsymbol{\theta} \boldsymbol{\theta}'.$$

where  $\mathbf{I}_p$  is the identity matrix of dimension  $p$ .

**Usage**

```
signs(X, theta, Gamma = NULL, check_X = FALSE)
```

```
cosines(X, theta, check_X = FALSE)
```

```
Gamma_theta(theta, eig = FALSE)
```

**Arguments**

X	hyperspherical data, a matrix of size $c(n, p)$ with unit-norm rows. NAs are allowed.
theta	a unit-norm vector of length $p$ . Normalized internally if it does not have unit norm (with a warning message).
Gamma	output from <code>Gamma_theta(theta = theta)</code> . If NULL (default), it is computed internally.
check_X	whether to check the unit norms on the rows of X. Defaults to FALSE for performance reasons.
eig	whether $\Gamma_\theta$ is to be found using an eigendecomposition of $\mathbf{I}_p - \theta\theta'$ (inefficient). Defaults to FALSE.

**Details**

Note that the projection matrix  $\Gamma_\theta$  is *not* unique. In particular, any completion of  $\theta$  to an orthonormal basis  $\{\theta, \mathbf{v}_1, \dots, \mathbf{v}_{p-1}\}$  gives a set of  $p-1$  orthonormal  $p$ -vectors  $\{\mathbf{v}_1, \dots, \mathbf{v}_{p-1}\}$  that conform the columns of  $\Gamma_\theta$ . If `eig = FALSE`, this approach is employed by rotating the canonical completion of  $\mathbf{e}_1 = (1, 0, \dots, 0)$ ,  $\{\mathbf{e}_2, \dots, \mathbf{e}_p\}$ , by the rotation matrix that rotates  $\mathbf{e}_1$  to  $\theta$ :

$$\mathbf{H}_\theta = (\theta + \mathbf{e}_1)(\theta + \mathbf{e}_1)' / (1 + \theta_1) - \mathbf{I}_p.$$

If `eig = TRUE`, then a much more expensive eigendecomposition of  $\Gamma_\theta\Gamma_\theta' = \mathbf{I}_p - \theta\theta'$  is performed for determining  $\{\mathbf{v}_1, \dots, \mathbf{v}_{p-1}\}$ .

If `signs` and `cosines` are called with X without unit norms in the rows, then the results will be spurious. Setting `check_X = TRUE` prevents this from happening.

**Value**

Depending on the function:

- `cosines`: a vector of length  $n$  with the cosines of X.
- `signs`: a matrix of size  $c(n, p-1)$  with the multivariate signs of X.
- `Gamma_theta`: a projection matrix  $\Gamma_\theta$  of size  $c(p, p-1)$ .

**Author(s)**

Eduardo García-Portugués, Davy Paindaveine, and Thomas Verdebout.

**References**

García-Portugués, E., Paindaveine, D., Verdebout, T. (2020) On optimal tests for rotational symmetry against new classes of hyperspherical distributions. *Journal of the American Statistical Association*, 115(532):1873–1887. doi: [10.1080/01621459.2019.1665527](https://doi.org/10.1080/01621459.2019.1665527)

**Examples**

```

# Gamma_theta
theta <- c(0, 1)
Gamma_theta(theta = theta)

# Signs and cosines for p = 2
L <- rbind(c(1, 0.5),
           c(0.5, 1))
X <- r_ACG(n = 1e3, Lambda = L)
par(mfrow = c(1, 2))
plot(signs(X = X, theta = theta), main = "Signs", xlab = expression(x[1]),
      ylab = expression(x[2]))
hist(cosines(X = X, theta = theta), prob = TRUE, main = "Cosines",
      xlab = expression(x * "'" * theta))

# Signs and cosines for p = 3
L <- rbind(c(2, 0.25, 0.25),
           c(0.25, 0.5, 0.25),
           c(0.25, 0.25, 0.5))
X <- r_ACG(n = 1e3, Lambda = L)
par(mfrow = c(1, 2))
theta <- c(0, 1, 0)
plot(signs(X = X, theta = theta), main = "Signs", xlab = expression(x[1]),
      ylab = expression(x[2]))
hist(cosines(X = X, theta = theta), prob = TRUE, main = "Cosines",
      xlab = expression(x * "'" * theta))

```

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 estimators

*Estimators for the axis of rotational symmetry  $\theta$* 


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**Description**

Estimation of the axis of rotational symmetry  $\theta$  of a rotational symmetric unit-norm random vector  $\mathbf{X}$  in  $S^{p-1} := \{\mathbf{x} \in R^p : \|\mathbf{x}\| = 1\}$ ,  $p \geq 2$ , from a hyperspherical sample  $\mathbf{X}_1, \dots, \mathbf{X}_n \in S^{p-1}$ .

**Usage**

```
spherical_mean(data)
```

```
spherical_loc_PCA(data)
```

**Arguments**

**data** hyperspherical data, a matrix of size  $c(n, p)$  with unit norm rows. Normalized internally if any row does not have unit norm (with a warning message). NAs are ignored.

## Details

The `spherical_mean` estimator computes the sample mean of  $\mathbf{X}_1, \dots, \mathbf{X}_n$  and normalizes it by its norm (if the norm is different from zero). It estimates consistently  $\boldsymbol{\theta}$  for rotational symmetric models based on [angular functions](#)  $g$  that are monotone increasing.

The estimator in `spherical_loc_PCA` is based on the fact that, under rotational symmetry, the expectation of  $\mathbf{X}\mathbf{X}'$  is  $a\boldsymbol{\theta}\boldsymbol{\theta}' + b(\mathbf{I}_p - \boldsymbol{\theta}\boldsymbol{\theta}')$  for certain constants  $a, b \geq 0$ . Therefore,  $\boldsymbol{\theta}$  is the eigenvector with unique multiplicity of the expectation of  $\mathbf{X}\mathbf{X}'$ . Its use is recommended if the rotationally symmetric data is not unimodal.

## Value

A vector of length  $p$  with an estimate for  $\boldsymbol{\theta}$ .

## Author(s)

Eduardo García-Portugués, Davy Paindaveine, and Thomas Verdebout.

## References

García-Portugués, E., Paindaveine, D., Verdebout, T. (2020) On optimal tests for rotational symmetry against new classes of hyperspherical distributions. *Journal of the American Statistical Association*, 115(532):1873–1887. doi: [10.1080/01621459.2019.1665527](https://doi.org/10.1080/01621459.2019.1665527)

## Examples

```
# Sample from a vMF
n <- 200
p <- 10
theta <- c(1, rep(0, p - 1))
set.seed(123456789)
data <- r_vMF(n = n, mu = theta, kappa = 3)
theta_mean <- spherical_mean(data)
theta_PCA <- spherical_loc_PCA(data)
sqrt(sum((theta - theta_mean)^2)) # More efficient
sqrt(sum((theta - theta_PCA)^2))

# Sample from a mixture of antipodal vMF's
n <- 200
p <- 10
theta <- c(1, rep(0, p - 1))
set.seed(123456789)
data <- rbind(r_vMF(n = n, mu = theta, kappa = 3),
             r_vMF(n = n, mu = -theta, kappa = 3))
theta_mean <- spherical_mean(data)
theta_PCA <- spherical_loc_PCA(data)
sqrt(sum((theta - theta_mean)^2))
sqrt(sum((theta - theta_PCA)^2)) # Better suited in this case
```

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sunspots\_births      *Recorded sunspots births during 1872–2018*

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### Description

Processed version of the **Debrecen Photoheliographic Data (DPD)** sunspot catalogue and the revised version of the **Greenwich Photoheliographic Results (GPR)** sunspot catalogue. The two sources contain the records of sunspots appeared during 1872–2018 (GPR for 1872–1976; DPD for 1974–2018).

Sunspots appear in groups and have a variable lifetime. This dataset has been processed to account only for the *births* or emergences (first observations) of *groups* of sunspots.

### Usage

sunspots\_births

### Format

A data frame with 51303 rows and 6 variables:

**date** UTC date, as **POSIXct**, of the first observation of a group of sunspots.

**cycle** **solar cycle** in which the group of sunspots was observed.

**total\_area** total whole spot area of the group, measured in millionths of the solar hemisphere.

**dist\_sun\_disc** distance from the center of Sun's disc, measured in units of the solar radius.

**theta** mean longitude angle  $\theta \in [0, 2\pi)$  of the group position.

**phi** mean latitude angle  $\phi \in [-\pi/2, \pi/2)$  of the group position.

### Details

The mean position of the group of sunspots is obtained by a weighted average of the positions of the single sunspots by the whole spot area of the single spots. The areas are corrected to account for foreshortening.

The  $(\theta, \phi)$  angles are such their associated Cartesian coordinates are:

$$(\cos(\phi) \cos(\theta), \cos(\phi) \sin(\theta), \sin(\phi)),$$

with  $(0, 0, 1)$  denoting the north pole.

The DPD data has **different states** of completeness and quality control. The longest span of "final complete data" (no missing observation days and the data has undergone a systematic quality control) is from 2005 to 2015.

The data has been preprocessed using the following pipeline:

1. Retrieve data from the GPR and DPD sunspot catalogues.
2. Omit observations with NAs in the sunspot positions.
3. Filter for sunspot groups.



4. Relabel the NOAA identifier for the sunspot group for records before 1974, prefixing the "GPR" string. Otherwise, very different groups of sunspots from the two catalogues may share the same identifier.
5. Keep only the first row of each NOAA instance, the first-ever observation of each sunspot group.

The script performing the preprocessing is available at [sunspots-births.R](#)

### Author(s)

Data processed by Eduardo García-Portugués, Davy Paindaveine, and Thomas Verdebout from the original sources.

### Source

<http://fenyi.solarobs.csfk.mta.hu>

### References

- Baranyi, T., Győri, L., Ludmány, A. (2016) On-line tools for solar data compiled at the Debrecen observatory and their extensions with the Greenwich sunspot data. *Solar Physics*, 291(9–10):3081–3102. doi: [10.1007/s1120701609301](https://doi.org/10.1007/s1120701609301)
- Győri, L., Ludmány, A., Baranyi, T. (2019) Comparative analysis of Debrecen sunspot catalogues. *Monthly Notices of the Royal Astronomical Society*, 465(2):1259–1273. doi: [10.1093/mnras/stw2667](https://doi.org/10.1093/mnras/stw2667)

### Examples

```
# Load data
data("sunspots_births")

# Transform to Cartesian coordinates
sunspots_births$X <-
  cbind(cos(sunspots_births$phi) * cos(sunspots_births$theta),
        cos(sunspots_births$phi) * sin(sunspots_births$theta),
        sin(sunspots_births$phi))

# Plot data associated to the 23rd cycle
sunspots_23 <- subset(sunspots_births, cycle == 23)
n <- nrow(sunspots_23$X)
if (requireNamespace("rgl")) {
  rgl::plot3d(0, 0, 0, xlim = c(-1, 1), ylim = c(-1, 1), zlim = c(-1, 1),
             radius = 1, type = "s", col = "lightblue", alpha = 0.25,
             lit = FALSE)
}
n_cols <- 100
cuts <- cut(x = sunspots_23$date, include.lowest = TRUE,
           breaks = quantile(sunspots_23$date,
                             probs = seq(0, 1, l = n_cols + 1)))
if (requireNamespace("rgl")) {
  rgl::points3d(sunspots_23$X, col = viridisLite::viridis(n_cols)[cuts])
}
```

```

# Spörer's law: sunspots at the beginning of the solar cycle (dark blue
# color) tend to appear at higher latitudes, gradually decreasing to the
# equator as the solar cycle advances (yellow color)

# Estimation of the density of the cosines
V <- cosines(X = sunspots_23$X, theta = c(0, 0, 1))
h <- bw.SJ(x = V, method = "dpi")
plot(kde <- density(x = V, bw = h, n = 2^13, from = -1, to = 1), col = 1,
      xlim = c(-1, 1), ylim = c(0, 3), axes = FALSE, main = "",
      xlab = "Cosines (latitude angles)", lwd = 2)
at <- seq(-1, 1, by = 0.25)
axis(2); axis(1, at = at)
axis(1, at = at, line = 1, tick = FALSE,
      labels = paste0("(", 90 - round(acos(at) / pi * 180, 1), "°)")
rug(V)
legend("topright", legend = c("Full cycle", "Initial 25% cycle",
                              "Final 25% cycle"),
      lwd = 2, col = c(1, viridisLite::viridis(12)[c(3, 8)]))

# Density for the observations within the initial 25% of the cycle
part1 <- sunspots_23$date < quantile(sunspots_23$date, 0.25)
V1 <- cosines(X = sunspots_23$X[part1, ], theta = c(0, 0, 1))
h1 <- bw.SJ(x = V1, method = "dpi")
lines(kde1 <- density(x = V1, bw = h1, n = 2^13, from = -1, to = 1),
      col = viridisLite::viridis(12)[3], lwd = 2)

# Density for the observations within the final 25% of the cycle
part2 <- sunspots_23$date > quantile(sunspots_23$date, 0.75)
V2 <- cosines(X = sunspots_23$X[part2, ], theta = c(0, 0, 1))
h2 <- bw.SJ(x = V2, method = "dpi")
lines(kde2 <- density(x = V2, bw = h2, n = 2^13, from = -1, to = 1),
      col = viridisLite::viridis(12)[8], lwd = 2)

# Computation the level set of a kernel density estimator that contains
# at least 1 - alpha of the probability (kde stands for an object
# containing the output of density(x = data))
kde_level_set <- function(kde, data, alpha) {

  # Estimate c from alpha
  c <- quantile(approx(x = kde$x, y = kde$y, xout = data)$y, probs = alpha)

  # Begin and end index for the potentially many intervals in the level sets
  kde_larger_c <- kde$y >= c
  run_length_kde <- rle(kde_larger_c)
  begin <- which(diff(kde_larger_c) > 0)
  end <- begin + run_length_kde$lengths[run_length_kde$values] - 1

  # Return the [a_i, b_i], i = 1, ..., K in the K rows
  return(cbind(kde$x[begin], kde$x[end]))

}

# Level set containing the 90% of the probability, in latitude angles

```

```

90 - acos(kde_level_set(kde = kde, data = V, alpha = 0.10)) / pi * 180

# Modes (in cosines and latitude angles)
modes <- c(kde$x[kde$x < 0][which.max(kde$y[kde$x < 0])],
          kde$x[kde$x > 0][which.max(kde$y[kde$x > 0])])
90 - acos(modes) / pi * 180

```

tang-norm-decomp

*Distributions based on the tangent-normal decomposition***Description**

Density and simulation of a distribution on  $S^{p-1} := \{\mathbf{x} \in R^p : \|\mathbf{x}\| = 1\}$ ,  $p \geq 2$ , obtained by the tangent-normal decomposition. The *tangent-normal decomposition* of the random vector  $\mathbf{X} \in S^{p-1}$  is

$$V\boldsymbol{\theta} + \sqrt{1 - V^2}\boldsymbol{\Gamma}_\theta\mathbf{U}$$

where  $V := \mathbf{X}'\boldsymbol{\theta}$  is a random variable in  $[-1, 1]$  (the *cosines* of  $\mathbf{X}$ ) and  $\mathbf{U} := \boldsymbol{\Gamma}_\theta\mathbf{X}/\|\boldsymbol{\Gamma}_\theta\mathbf{X}\|$  is a random vector in  $S^{p-2}$  (the *multivariate signs* of  $\mathbf{X}$ ) and  $\boldsymbol{\Gamma}_\theta$  is the  $p \times (p-1)$  matrix computed by [Gamma\\_theta](#).

The tangent-normal decomposition can be employed for constructing distributions for  $\mathbf{X}$  that arise for certain choices of  $V$  and  $\mathbf{U}$ . If  $V$  and  $\mathbf{U}$  are *independent*, then simulation from  $\mathbf{X}$  is straightforward using the tangent-normal decomposition. Also, the density of  $\mathbf{X}$  at  $\mathbf{x} \in S^{p-1}$ ,  $f_{\mathbf{X}}(\mathbf{x})$ , is readily computed as

$$f_{\mathbf{X}}(\mathbf{x}) = \omega_{p-1}c_g g(t)(1 - t^2)^{(p-3)/2}f_{\mathbf{U}}(\mathbf{u})$$

where  $t := \mathbf{x}'\boldsymbol{\theta}$ ,  $\mathbf{u} := \boldsymbol{\Gamma}_\theta\mathbf{x}/\|\boldsymbol{\Gamma}_\theta\mathbf{x}\|$ ,  $f_{\mathbf{U}}$  is the density of  $\mathbf{U}$ , and  $f_V(v) := \omega_{p-1}c_g g(v)(1 - v^2)^{(p-3)/2}$  is the density of  $V$  for an angular function  $g$  with normalizing constant  $c_g$ .  $\omega_{p-1}$  is the surface area of  $S^{p-2}$ .

**Usage**

```
d_tang_norm(x, theta, g_scaled, d_V, d_U, log = FALSE)
```

```
r_tang_norm(n, theta, r_U, r_V)
```

**Arguments**

<code>x</code>	locations in $S^{p-1}$ to evaluate the density. Either a matrix of size <code>c(nx, p)</code> or a vector of length <code>p</code> . Normalized internally if required (with a warning message).
<code>theta</code>	a unit norm vector of size <code>p</code> giving the axis of rotational symmetry.
<code>g_scaled</code>	the <i>scaled</i> angular density $c_g g$ . In the form <code>g_scaled &lt;-function(t, log = TRUE) { ... }</code> . See examples.
<code>d_V</code>	the density $f_V$ . In the form <code>d_V &lt;-function(v, log = TRUE) { ... }</code> . See examples.
<code>d_U</code>	the density $f_{\mathbf{U}}$ . In the form <code>d_U &lt;-function(u, log = TRUE) { ... }</code> . See examples.

log	flag to indicate if the logarithm of the density (or the normalizing constant) is to be computed.
n	sample size, a positive integer.
r_U	a function for simulating $U$ . Its first argument must be the sample size. See examples.
r_V	a function for simulating $V$ . Its first argument must be the sample size. See examples.

### Details

Either `g_scaled` or `d_V` can be supplied to `d_tang_norm` (the rest of the arguments are compulsory). One possible choice for `g_scaled` is `g_vmf` with `scaled = TRUE`. Another possible choice is the angular function  $g(t) = 1 - t^2$ , normalized by its normalizing constant  $c_g = (\Gamma(p/2)p)/(2\pi^{p/2}(p-1))$  (see examples). This angular function makes  $V^2$  to be distributed as a  $\text{Beta}(1/2, (p+1)/2)$ .

The normalizing constants and densities are computed through log-scales for numerical accuracy.

### Value

Depending on the function:

- `d_tang_norm`: a vector of length `nx` or 1 with the evaluated density at `x`.
- `r_tang_norm`: a matrix of size `c(n, p)` with the random sample.

### Author(s)

Eduardo García-Portugués, Davy Paindaveine, and Thomas Verdebout.

### References

García-Portugués, E., Paindaveine, D., Verdebout, T. (2020) On optimal tests for rotational symmetry against new classes of hyperspherical distributions. *Journal of the American Statistical Association*, 115(532):1873–1887. doi: [10.1080/01621459.2019.1665527](https://doi.org/10.1080/01621459.2019.1665527)

### See Also

[Gamma\\_theta](#), [signs](#), [tangent-elliptical](#), [tangent-vmf](#), [vmf](#).

### Examples

```
## Simulation and density evaluation for p = 2

# Parameters
n <- 1e3
p <- 2
theta <- c(rep(0, p - 1), 1)
mu <- c(rep(0, p - 2), 1)
kappa_V <- 2
kappa_U <- 0.1
```

```

# The vMF scaled angular function
g_scaled <- function(t, log) {
  g_vMF(t, p = p - 1, kappa = kappa_V, scaled = TRUE, log = log)
}

# Cosine density for the vMF distribution
d_V <- function(v, log) {
  log_dens <- g_scaled(v, log = log) + (p - 3)/2 * log(1 - v^2)
  switch(log + 1, exp(log_dens), log_dens)
}

# Multivariate signs density based on a vMF
d_U <- function(x, log) d_vMF(x = x, mu = mu, kappa = kappa_U, log = log)

# Simulation functions
r_V <- function(n) r_g_vMF(n = n, p = p, kappa = kappa_V)
r_U <- function(n) r_vMF(n = n, mu = mu, kappa = kappa_U)

# Sample and color according to density
x <- r_tang_norm(n = n, theta = theta, r_V = r_V, r_U = r_U)
r <- runif(n, 0.95, 1.05) # Radius perturbation to improve visualization
col <- viridisLite::viridis(n)
dens <- d_tang_norm(x = x, theta = theta, g_scaled = g_scaled, d_U = d_U)
# dens <- d_tang_norm(x = x, theta = theta, d_V = d_V, d_U = d_U) # The same
plot(r * x, pch = 16, col = col[rank(dens)])

## Simulation and density evaluation for p = 3

# Parameters
p <- 3
n <- 5e3
theta <- c(rep(0, p - 1), 1)
mu <- c(rep(0, p - 2), 1)
kappa_V <- 2
kappa_U <- 2

# Sample and color according to density
x <- r_tang_norm(n = n, theta = theta, r_V = r_V, r_U = r_U)
col <- viridisLite::viridis(n)
dens <- d_tang_norm(x = x, theta = theta, g_scaled = g_scaled, d_U = d_U)
if (requireNamespace("rgl")) {
  rgl::plot3d(x, col = col[rank(dens)], size = 5)
}

## A non-vMF angular function:  $g(t) = 1 - t^2$ . It is associated to the
## Beta(1/2, (p + 1)/2) distribution.

# Scaled angular function
g_scaled <- function(t, log) {
  log_c_g <- lgamma(0.5 * p) + log(0.5 * p / (p - 1)) - 0.5 * p * log(pi)
  log_g <- log_c_g + log(1 - t^2)
  switch(log + 1, exp(log_g), log_g)
}

```

```

# Cosine density
d_V <- function(v, log) {
  log_dens <- w_p(p = p - 1, log = TRUE) + g_scaled(t = v, log = TRUE) +
    (0.5 * (p - 3)) * log(1 - v^2)
  switch(log + 1, exp(log_dens), log_dens)
}

# Simulation
r_V <- function(n) {
  sample(x = c(-1, 1), size = n, replace = TRUE) *
    sqrt(rbeta(n = n, shape1 = 0.5, shape2 = 0.5 * (p + 1)))
}

# Sample and color according to density
r_U <- function(n) r_unif_sphere(n = n, p = p - 1)
x <- r_tang_norm(n = n, theta = theta, r_V = r_V, r_U = r_U)
col <- viridisLite::viridis(n)
dens <- d_tang_norm(x = x, theta = theta, d_V = d_V, d_U = d_unif_sphere)
# dens <- d_tang_norm(x = x, theta = theta, g_scaled = g_scaled,
#                   d_U = d_unif_sphere) # The same
if (requireNamespace("rgl")) {
  rgl::plot3d(x, col = col[rank(dens)], size = 5)
}

```

---

tangent-elliptical      *Tangent elliptical distribution*

---

## Description

Density and simulation of the Tangent Elliptical (TE) distribution on  $S^{p-1} := \{\mathbf{x} \in R^p : \|\mathbf{x}\| = 1\}$ ,  $p \geq 2$ . The distribution arises by considering the [tangent-normal decomposition](#) with multivariate [signs](#) distributed as an [Angular Central Gaussian](#) distribution.

## Usage

```
d_TE(x, theta, g_scaled, d_V, Lambda, log = FALSE)
```

```
r_TE(n, theta, r_V, Lambda)
```

## Arguments

x	locations in $S^{p-1}$ to evaluate the density. Either a matrix of size $c(nx, p)$ or a vector of length $p$ . Normalized internally if required (with a warning message).
theta	a unit norm vector of size $p$ giving the axis of rotational symmetry.
g_scaled	the <i>scaled</i> angular density $c_{gg}$ . In the form <code>g_scaled &lt;- function(t, log = TRUE) { ... }</code> . See examples.
d_V	the density $f_V$ . In the form <code>d_V &lt;- function(v, log = TRUE) { ... }</code> . See examples.

Lambda	the shape matrix $\Lambda$ of the ACG used in the multivariate signs. A symmetric and positive definite matrix of size $c(p - 1, p - 1)$ .
log	flag to indicate if the logarithm of the density (or the normalizing constant) is to be computed.
n	sample size, a positive integer.
r_V	a function for simulating $V$ . Its first argument must be the sample size. See examples.

### Details

The functions are wrappers for [d\\_tang\\_norm](#) and [r\\_tang\\_norm](#) with  $d_U = d\_ACG$  and  $r_U = r\_ACG$ .

### Value

Depending on the function:

- $d_{TE}$ : a vector of length  $nx$  or 1 with the evaluated density at  $x$ .
- $r_{TE}$ : a matrix of size  $c(n, p)$  with the random sample.

### Author(s)

Eduardo García-Portugués, Davy Paindaveine, and Thomas Verdebout.

### References

García-Portugués, E., Paindaveine, D., Verdebout, T. (2020) On optimal tests for rotational symmetry against new classes of hyperspherical distributions. *Journal of the American Statistical Association*, 115(532):1873–1887. doi: [10.1080/01621459.2019.1665527](https://doi.org/10.1080/01621459.2019.1665527)

### See Also

[tang-norm-decomp](#), [tangent-vmf](#), [ACG](#).

### Examples

```
## Simulation and density evaluation for p = 2

# Parameters
p <- 2
n <- 1e3
theta <- c(rep(0, p - 1), 1)
Lambda <- matrix(0.5, nrow = p - 1, ncol = p - 1)
diag(Lambda) <- 1
kappa_V <- 2

# Required functions
r_V <- function(n) r_g_VMF(n = n, p = p, kappa = kappa_V)
g_scaled <- function(t, log) {
  g_VMF(t, p = p - 1, kappa = kappa_V, scaled = TRUE, log = log)
}
```

```

# Sample and color according to density
x <- r_TE(n = n, theta = theta, r_V = r_V, Lambda = Lambda)
col <- viridisLite::viridis(n)
r <- runif(n, 0.95, 1.05) # Radius perturbation to improve visualization
dens <- d_TE(x = x, theta = theta, g_scaled = g_scaled, Lambda = Lambda)
plot(r * x, pch = 16, col = col[rank(dens)])

## Simulation and density evaluation for p = 3

# Parameters
p <- 3
n <- 5e3
theta <- c(rep(0, p - 1), 1)
Lambda <- matrix(0.5, nrow = p - 1, ncol = p - 1)
diag(Lambda) <- 1
kappa_V <- 2

# Sample and color according to density
x <- r_TE(n = n, theta = theta, r_V = r_V, Lambda = Lambda)
col <- viridisLite::viridis(n)
dens <- d_TE(x = x, theta = theta, g_scaled = g_scaled, Lambda = Lambda)
if (requireNamespace("rgl")) {
  rgl::plot3d(x, col = col[rank(dens)], size = 5)
}

## A non-vMF angular function:  $g(t) = 1 - t^2$ . It is associated to the
## Beta(1/2, (p + 1)/2) distribution.

# Scaled angular function
g_scaled <- function(t, log) {
  log_c_g <- lgamma(0.5 * p) + log(0.5 * p / (p - 1)) - 0.5 * p * log(pi)
  log_g <- log_c_g + log(1 - t^2)
  switch(log + 1, exp(log_g), log_g)
}

# Simulation
r_V <- function(n) {
  sample(x = c(-1, 1), size = n, replace = TRUE) *
  sqrt(rbeta(n = n, shape1 = 0.5, shape2 = 0.5 * (p + 1)))
}

# Sample and color according to density
kappa_V <- 1
Lambda <- matrix(0.75, nrow = p - 1, ncol = p - 1)
diag(Lambda) <- 1
x <- r_TE(n = n, theta = theta, r_V = r_V, Lambda = Lambda)
col <- viridisLite::viridis(n)
dens <- d_TE(x = x, theta = theta, g_scaled = g_scaled, Lambda = Lambda)
if (requireNamespace("rgl")) {
  rgl::plot3d(x, col = col[rank(dens)], size = 5)
}

```



---

tangent-vMF	<i>Tangent von Mises–Fisher distribution</i>
-------------	--

---

### Description

Density and simulation of the Tangent von Mises–Fisher (TM) distribution on  $S^{p-1} := \{\mathbf{x} \in \mathbb{R}^p : \|\mathbf{x}\| = 1\}$ ,  $p \geq 2$ . The distribution arises by considering the [tangent-normal decomposition](#) with multivariate [signs](#) distributed as a [von Mises–Fisher](#) distribution.

### Usage

```
d_TM(x, theta, g_scaled, d_V, mu, kappa, log = FALSE)
```

```
r_TM(n, theta, r_V, mu, kappa)
```

### Arguments

x	locations in $S^{p-1}$ to evaluate the density. Either a matrix of size $c(\text{nx}, p)$ or a vector of length $p$ . Normalized internally if required (with a warning message).
theta	a unit norm vector of size $p$ giving the axis of rotational symmetry.
g_scaled	the <i>scaled</i> angular density $c_g g$ . In the form <code>g_scaled &lt;- function(t, log = TRUE) { ... }</code> . See examples.
d_V	the density $f_V$ . In the form <code>d_V &lt;- function(v, log = TRUE) { ... }</code> . See examples.
mu	the directional mean $\mu$ of the vMF used in the multivariate signs. A unit-norm vector of length $p - 1$ .
kappa	concentration parameter $\kappa$ of the vMF used in the multivariate signs. A nonnegative scalar.
log	flag to indicate if the logarithm of the density (or the normalizing constant) is to be computed.
n	sample size, a positive integer.
r_V	a function for simulating $V$ . Its first argument must be the sample size. See examples.

### Details

The functions are wrappers for [d\\_tang\\_norm](#) and [r\\_tang\\_norm](#) with `d_U = d_vMF` and `r_U = r_vMF`.

### Value

Depending on the function:

- `d_TM`: a vector of length `nx` or 1 with the evaluated density at `x`.
- `r_TM`: a matrix of size  $c(n, p)$  with the random sample.

**Author(s)**

Eduardo García-Portugués, Davy Paindaveine, and Thomas Verdebout.

**References**

García-Portugués, E., Paindaveine, D., Verdebout, T. (2020) On optimal tests for rotational symmetry against new classes of hyperspherical distributions. *Journal of the American Statistical Association*, 115(532):1873–1887. doi: [10.1080/01621459.2019.1665527](https://doi.org/10.1080/01621459.2019.1665527)

**See Also**

[tang-norm-decomp](#), [tangent-elliptical](#), [VMF](#).

**Examples**

```
## Simulation and density evaluation for p = 2

# Parameters
p <- 2
n <- 1e3
theta <- c(rep(0, p - 1), 1)
mu <- c(rep(0, p - 2), 1)
kappa <- 1
kappa_V <- 2

# Required functions
r_V <- function(n) r_g_VMF(n = n, p = p, kappa = kappa_V)
g_scaled <- function(t, log) {
  g_VMF(t, p = p - 1, kappa = kappa_V, scaled = TRUE, log = log)
}

# Sample and color according to density
x <- r_TM(n = n, theta = theta, r_V = r_V, mu = 1, kappa = kappa)
col <- viridisLite::viridis(n)
r <- runif(n, 0.95, 1.05) # Radius perturbation to improve visualization
dens <- d_TM(x = x, theta = theta, g_scaled = g_scaled, mu = mu,
            kappa = kappa)
plot(r * x, pch = 16, col = col[rank(dens)])

## Simulation and density evaluation for p = 3

# Parameters
p <- 3
n <- 5e3
theta <- c(rep(0, p - 1), 1)
mu <- c(rep(0, p - 2), 1)
kappa <- 1
kappa_V <- 2

# Sample and color according to density
x <- r_TM(n = n, theta = theta, r_V = r_V, mu = mu, kappa = kappa)
col <- viridisLite::viridis(n)
```

```

dens <- d_TM(x = x, theta = theta, g_scaled = g_scaled, mu = mu,
            kappa = kappa)
if (requireNamespace("rgl")) {
  rgl::plot3d(x, col = col[rank(dens)], size = 5)
}

## A non-vMF angular function: g(t) = 1 - t^2. It is associated to the
## Beta(1/2, (p + 1)/2) distribution.

# Scaled angular function
g_scaled <- function(t, log) {
  log_c_g <- lgamma(0.5 * p) + log(0.5 * p / (p - 1)) - 0.5 * p * log(pi)
  log_g <- log_c_g + log(1 - t^2)
  switch(log + 1, exp(log_g), log_g)
}

# Simulation
r_V <- function(n) {
  sample(x = c(-1, 1), size = n, replace = TRUE) *
  sqrt(rbeta(n = n, shape1 = 0.5, shape2 = 0.5 * (p + 1)))
}

# Sample and color according to density
kappa <- 0.5
x <- r_TM(n = n, theta = theta, r_V = r_V, mu = mu, kappa = kappa)
col <- viridisLite::viridis(n)
dens <- d_TM(x = x, theta = theta, g_scaled = g_scaled,
            mu = mu, kappa = kappa)
if (requireNamespace("rgl")) {
  rgl::plot3d(x, col = col[rank(dens)], size = 5)
}

```

---

test\_rotasym

*Tests of rotational symmetry for hyperspherical data*


---

### Description

Tests for assessing the rotational symmetry of a unit-norm random vector  $\mathbf{X}$  in  $S^{p-1} := \{\mathbf{x} \in R^p : \|\mathbf{x}\| = 1\}$ ,  $p \geq 2$ , about a location  $\boldsymbol{\theta} \in S^{p-1}$ , from a hyperspherical sample  $\mathbf{X}_1, \dots, \mathbf{X}_n \in S^{p-1}$ .

The vector  $\mathbf{X}$  is said to be rotational symmetric about  $\boldsymbol{\theta}$  if the distributions of  $\mathbf{O}\mathbf{X}$  and  $\mathbf{X}$  coincide, where  $\mathbf{O}$  is any  $p \times p$  rotation matrix that fixes  $\boldsymbol{\theta}$ , i.e.,  $\mathbf{O}\boldsymbol{\theta} = \boldsymbol{\theta}$ .

### Usage

```

test_rotasym(data, theta = spherical_mean, type = c("sc", "loc", "loc_vMF",
          "hyb", "hyb_vMF")[5], Fisher = FALSE, U = NULL, V = NULL)

```

## Arguments

data	hyperspherical data, a matrix of size $c(n, p)$ with unit norm rows. Normalized internally if any row does not have unit norm (with a warning message). NAs are ignored.
theta	either a unit norm vector of size $p$ giving the axis of rotational symmetry (for the specified- $\theta$ case) or a function that implements an estimator $\hat{\theta}$ of $\theta$ (for the unspecified- $\theta$ case). The default calls the <code>spherical_mean</code> function. See examples.
type	<p>a character string (case insensitive) indicating the type of test to conduct:</p> <ul style="list-style-type: none"> <li>• "sc": "scatter" test based on the statistic <math>Q_{\theta}^{\text{sc}}</math>. Evaluates if the covariance matrix of the multivariate signs is isotropic.</li> <li>• "loc": "location" test based on the statistic <math>Q_{\theta}^{\text{loc}}</math>. Evaluates if the expectation of the multivariate signs is zero.</li> <li>• "loc_vMF": adapted "location" test, based on the statistic <math>Q_{\text{vMF}}^{\text{loc}}</math>.</li> <li>• "hyb": "hybrid" test based on the statistics <math>Q_{\theta}^{\text{sc}}</math> and <math>Q_{\theta}^{\text{loc}}</math>.</li> <li>• "hyb_vMF" (default): adapted "hybrid" test based on the statistics <math>Q_{\theta}^{\text{sc}}</math> and <math>Q_{\text{vMF}}^{\text{loc}}</math>.</li> </ul> <p>See the details below for further explanations of the tests.</p>
Fisher	if TRUE, then Fisher's method is employed to aggregate the scatter and location tests in the hybrid test, see details below. Otherwise, the hybrid statistic is the sum of the scatter and location statistics. Defaults to FALSE.
U	<i>multivariate signs</i> of data, a matrix of size $c(n, p - 1)$ . Computed if NULL (the default).
V	<i>cosines</i> of data, a vector of size $n$ . Computed if NULL (the default).

## Details

Descriptions of the tests:

- The "scatter" test is locally and asymptotically optimal against [tangent elliptical](#) alternatives to rotational symmetry. However, it is not consistent against [tangent von Mises–Fisher \(vMF\)](#) alternatives. The asymptotic null distribution of  $Q_{\theta}^{\text{sc}}$  is unaffected if  $\theta$  is estimated, that is, the asymptotic null distributions of  $Q_{\theta}^{\text{sc}}$  and  $Q_{\hat{\theta}}^{\text{sc}}$  are the same.
- The "location" test is locally and asymptotically most powerful against vMF alternatives to rotational symmetry. However, it is not consistent against tangent elliptical alternatives. The asymptotic null distribution of  $Q_{\theta}^{\text{loc}}$  for known  $\theta$  (the one implemented in `test_rotasym`) *does change* if  $\theta$  is estimated by  $\hat{\theta}$ . Therefore, if the test is performed with an estimated  $\theta$  (if theta is a function)  $Q_{\hat{\theta}}^{\text{loc}}$  will not be properly calibrated. `test_rotasym` will give a warning in such case.
- The "vMF location" test is a modification of the "location" test designed to make its null asymptotic distribution invariant from the estimation of  $\theta$  (as the "scatter" test is). The test is optimal against tangent vMF alternatives with a *specific*, vMF-based, angular function `g_vMF`. Despite not being optimal against all tangent vMF alternatives, it is consistent for all of them. As the location test, it is not consistent against tangent elliptical alternatives.

- The "hybrid" test combines (see below how) the "scatter" and "location" tests. The test is neither optimal against tangent elliptical nor tangent vMF alternatives, but it is consistent against both. Since it is based on the "location" test, if computed with an estimator  $\hat{\theta}$ , the test statistic will not be properly calibrated. `test_rotasym` will give a warning in such case.
- The "vMF hybrid" test is the analogous of the "hybrid" test but replaces the "location" test by the "vMF location" test.

The combination of the scatter and location tests in the hybrid tests is done in two different ways:

- If `Fisher = FALSE`, then the scatter and location tests statistics give the hybrid test statistic

$$Q_{\theta}^{\text{hyb}} := Q_{\theta}^{\text{sc}} + Q_{\theta}^{\text{loc}}.$$

- If `Fisher = TRUE`, then Fisher's method for aggregating independent tests (the two test statistics are independent under rotational symmetry) is considered, resulting the hybrid test statistic:

$$Q_{\theta}^{\text{hyb}} := -2(\log(p_{\text{sc}}) + \log(p_{\text{loc}}))$$

where  $p_{\text{sc}}$  and  $p_{\text{loc}}$  are the  $p$ -values of the scatter and location tests, respectively.

The hybrid test statistic  $Q_{\text{vMF}}^{\text{hyb}}$  follows analogously to  $Q_{\theta}^{\text{hyb}}$  by replacing  $Q_{\theta}^{\text{loc}}$  with  $Q_{\text{vMF}}^{\text{loc}}$ .

Finally, recall that the tests are designed to test *implications* of rotational symmetry. Therefore, the tests are not consistent against *all* types of alternatives to rotational symmetry.

## Value

An object of the `htest` class with the following elements:

- `statistic`: test statistic.
- `parameter`: degrees of freedom of the chi-square distribution appearing in all the null asymptotic distributions.
- `p.value`:  $p$ -value of the test.
- `method`: information on the type of test performed.
- `data.name`: name of the value of data.
- `U`: multivariate signs of data.
- `V`: cosines of data.

## Author(s)

Eduardo García-Portugués, Davy Paindaveine, and Thomas Verdebout.

## References

García-Portugués, E., Paindaveine, D., Verdebout, T. (2020) On optimal tests for rotational symmetry against new classes of hyperspherical distributions. *Journal of the American Statistical Association*, 115(532):1873–1887. doi: [10.1080/01621459.2019.1665527](https://doi.org/10.1080/01621459.2019.1665527)

## See Also

[tangent-elliptical](#), [tangent-vMF](#), [spherical\\_mean](#).

**Examples**

```

## Rotational symmetry holds

# Sample data from a vMF (rotational symmetric distribution about mu)
n <- 200
p <- 10
theta <- c(1, rep(0, p - 1))
set.seed(123456789)
data_0 <- r_vMF(n = n, mu = theta, kappa = 1)

# theta known
test_rotasym(data = data_0, theta = theta, type = "sc")
test_rotasym(data = data_0, theta = theta, type = "loc")
test_rotasym(data = data_0, theta = theta, type = "loc_vMF")
test_rotasym(data = data_0, theta = theta, type = "hyb")
test_rotasym(data = data_0, theta = theta, type = "hyb", Fisher = TRUE)
test_rotasym(data = data_0, theta = theta, type = "hyb_vMF")
test_rotasym(data = data_0, theta = theta, type = "hyb_vMF", Fisher = TRUE)

# theta unknown (employs the spherical mean as estimator)
test_rotasym(data = data_0, type = "sc")
test_rotasym(data = data_0, type = "loc") # Warning
test_rotasym(data = data_0, type = "loc_vMF")
test_rotasym(data = data_0, type = "hyb") # Warning
test_rotasym(data = data_0, type = "hyb", Fisher = TRUE) # Warning
test_rotasym(data = data_0, type = "hyb_vMF")
test_rotasym(data = data_0, type = "hyb_vMF", Fisher = TRUE)

## Rotational symmetry does not hold

# Sample non-rotational symmetric data from a tangent-vMF distribution
# The scatter test is blind to these deviations, while the location tests
# are optimal
n <- 200
p <- 10
theta <- c(1, rep(0, p - 1))
mu <- c(rep(0, p - 2), 1)
kappa <- 2
set.seed(123456789)
r_V <- function(n) {
  r_g_vMF(n = n, p = p, kappa = 1)
}
data_1 <- r_TM(n = n, r_V = r_V, theta = theta, mu = mu, kappa = kappa)

# theta known
test_rotasym(data = data_1, theta = theta, type = "sc")
test_rotasym(data = data_1, theta = theta, type = "loc")
test_rotasym(data = data_1, theta = theta, type = "loc_vMF")
test_rotasym(data = data_1, theta = theta, type = "hyb")
test_rotasym(data = data_1, theta = theta, type = "hyb", Fisher = TRUE)
test_rotasym(data = data_1, theta = theta, type = "hyb_vMF")
test_rotasym(data = data_1, theta = theta, type = "hyb_vMF", Fisher = TRUE)

```

```

# theta unknown (employs the spherical mean as estimator)
test_rotasym(data = data_1, type = "sc")
test_rotasym(data = data_1, type = "loc") # Warning
test_rotasym(data = data_1, type = "loc_vMF")
test_rotasym(data = data_1, type = "hyb") # Warning
test_rotasym(data = data_1, type = "hyb", Fisher = TRUE) # Warning
test_rotasym(data = data_1, type = "hyb_vMF")
test_rotasym(data = data_1, type = "hyb_vMF", Fisher = TRUE)

# Sample non-rotational symmetric data from a tangent-elliptical distribution
# The location tests are blind to these deviations, while the
# scatter test is optimal
n <- 200
p <- 10
theta <- c(1, rep(0, p - 1))
Lambda <- matrix(0.5, nrow = p - 1, ncol = p - 1)
diag(Lambda) <- 1
set.seed(123456789)
r_V <- function(n) {
  r_g_vMF(n = n, p = p, kappa = 1)
}
data_2 <- r_TE(n = n, r_V = r_V, theta = theta, Lambda = Lambda)

# theta known
test_rotasym(data = data_2, theta = theta, type = "sc")
test_rotasym(data = data_2, theta = theta, type = "loc")
test_rotasym(data = data_2, theta = theta, type = "loc_vMF")
test_rotasym(data = data_2, theta = theta, type = "hyb")
test_rotasym(data = data_2, theta = theta, type = "hyb", Fisher = TRUE)
test_rotasym(data = data_2, theta = theta, type = "hyb_vMF")
test_rotasym(data = data_2, theta = theta, type = "hyb_vMF", Fisher = TRUE)

# theta unknown (employs the spherical mean as estimator)
test_rotasym(data = data_2, type = "sc")
test_rotasym(data = data_2, type = "loc") # Warning
test_rotasym(data = data_2, type = "loc_vMF")
test_rotasym(data = data_2, type = "hyb") # Warning
test_rotasym(data = data_2, type = "hyb", Fisher = TRUE) # Warning
test_rotasym(data = data_2, type = "hyb_vMF")
test_rotasym(data = data_2, type = "hyb_vMF", Fisher = TRUE)

## Sunspots births data

# Load data
data("sunspots_births")
sunspots_births$X <-
  cbind(cos(sunspots_births$phi) * cos(sunspots_births$theta),
        cos(sunspots_births$phi) * sin(sunspots_births$theta),
        sin(sunspots_births$phi))

# Test rotational symmetry for the 23rd cycle, specified theta
sunspots_23 <- subset(sunspots_births, cycle == 23)

```

```

test_rotasym(data = sunspots_23$X, type = "sc", theta = c(0, 0, 1))
test_rotasym(data = sunspots_23$X, type = "loc", theta = c(0, 0, 1))
test_rotasym(data = sunspots_23$X, type = "hyb", theta = c(0, 0, 1))

# Test rotational symmetry for the 23rd cycle, unspecified theta
spherical_loc_PCA(sunspots_23$X)
test_rotasym(data = sunspots_23$X, type = "sc", theta = spherical_loc_PCA)
test_rotasym(data = sunspots_23$X, type = "loc_vMF",
              theta = spherical_loc_PCA)
test_rotasym(data = sunspots_23$X, type = "hyb_vMF",
              theta = spherical_loc_PCA)

# Test rotational symmetry for the 22nd cycle, specified theta
sunspots_22 <- subset(sunspots_births, cycle == 22)
test_rotasym(data = sunspots_22$X, type = "sc", theta = c(0, 0, 1))
test_rotasym(data = sunspots_22$X, type = "loc", theta = c(0, 0, 1))
test_rotasym(data = sunspots_22$X, type = "hyb", theta = c(0, 0, 1))

# Test rotational symmetry for the 22nd cycle, unspecified theta
spherical_loc_PCA(sunspots_22$X)
test_rotasym(data = sunspots_22$X, type = "sc", theta = spherical_loc_PCA)
test_rotasym(data = sunspots_22$X, type = "loc_vMF",
              theta = spherical_loc_PCA)
test_rotasym(data = sunspots_22$X, type = "hyb_vMF",
              theta = spherical_loc_PCA)

```

---

unif

*Uniform distribution on the hypersphere*


---

### Description

Density and simulation of the uniform distribution on  $S^{p-1} := \{\mathbf{x} \in R^p : \|\mathbf{x}\| = 1\}$ ,  $p \geq 1$ . The density is just the inverse of the surface area of  $S^{p-1}$ , given by

$$\omega_p := 2\pi^{p/2} / \Gamma(p/2).$$

### Usage

```
d_unif_sphere(x, log = FALSE)
```

```
r_unif_sphere(n, p)
```

```
w_p(p, log = FALSE)
```

### Arguments

**x** locations in  $S^{p-1}$  to evaluate the density. Either a matrix of size  $c(n_x, p)$  or a vector of length  $p$ . Normalized internally if required (with a warning message).

**log** flag to indicate if the logarithm of the density (or the normalizing constant) is to be computed.



$n$  sample size, a positive integer.  
 $p$  dimension of the ambient space  $R^p$  that contains  $S^{p-1}$ . A positive integer.

### Details

If  $p = 1$ , then  $S^0 = \{-1, 1\}$  and the "surface area" is 2. The function `w_p` is vectorized on  $p$ .

### Value

Depending on the function:

- `d_unif_sphere`: a vector of length `nx` or 1 with the evaluated density at `x`.
- `r_unif_sphere`: a matrix of size `c(n,p)` with the random sample.
- `w_p`: the surface area of  $S^{p-1}$ .

### Author(s)

Eduardo García-Portugués, Davy Paindaveine, and Thomas Verdebout.

### Examples

```
## Area of S^{p - 1}

# Areas of S^0, S^1, and S^2
w_p(p = 1:3)

# Area as a function of p
p <- 1:20
plot(p, w_p(p = p), type = "o", pch = 16, xlab = "p", ylab = "Area",
      main = expression("Surface area of " * S^{p - 1}), axes = FALSE)
box()
axis(1, at = p)
axis(2, at = seq(0, 34, by = 2))

## Simulation and density evaluation for p = 1, 2, 3

# p = 1
n <- 500
x <- r_unif_sphere(n = n, p = 1)
barplot(table(x) / n)
head(d_unif_sphere(x))

# p = 2
x <- r_unif_sphere(n = n, p = 3)
plot(x)
head(d_unif_sphere(x))

# p = 3
x <- r_unif_sphere(n = n, p = 3)
if (requireNamespace("rgl")) {
  rgl::plot3d(x)
```

```
}
head(d_unif_sphere(x))
```

vMF

*von Mises–Fisher distribution***Description**

Density and simulation of the von Mises–Fisher (vMF) distribution on  $S^{p-1} := \{\mathbf{x} \in R^p : \|\mathbf{x}\| = 1\}$ ,  $p \geq 1$ . The density at  $\mathbf{x} \in S^{p-1}$  is given by

$$c_{p,\kappa}^{\text{vMF}} e^{\kappa \mathbf{x}' \boldsymbol{\mu}} \quad \text{with} \quad c_{p,\kappa}^{\text{vMF}} := \kappa^{(p-2)/2} / ((2\pi)^{p/2} I_{(p-2)/2}(\kappa))$$

where  $\boldsymbol{\mu} \in S^{p-1}$  is the directional mean,  $\kappa \geq 0$  is the concentration parameter about  $\boldsymbol{\mu}$ , and  $I_\nu$  is the order- $\nu$  modified Bessel function of the first kind.

The angular function of the vMF is  $g(t) := e^{\kappa t}$ . The associated *cosines* density is  $\tilde{g}(v) := \omega_{p-1} c_{p,\kappa}^{\text{vMF}} g(v) (1-v^2)^{(p-3)/2}$ , where  $\omega_{p-1}$  is the surface area of  $S^{p-2}$ .

**Usage**

```
d_vMF(x, mu, kappa, log = FALSE)

c_vMF(p, kappa, log = FALSE)

r_vMF(n, mu, kappa)

g_vMF(t, p, kappa, scaled = TRUE, log = FALSE)

r_g_vMF(n, p, kappa)
```

**Arguments**

x	locations in $S^{p-1}$ to evaluate the density. Either a matrix of size $c(\text{nx}, p)$ or a vector of length $p$ . Normalized internally if required (with a warning message).
mu	the directional mean $\boldsymbol{\mu}$ of the vMF. A unit-norm vector of length $p$ .
kappa	concentration parameter $\kappa$ of the vMF. A nonnegative scalar. Can be a vector for <code>c_vMF</code> .
log	flag to indicate if the logarithm of the density (or the normalizing constant) is to be computed.
p	dimension of the ambient space $R^p$ that contains $S^{p-1}$ . A positive integer.
n	sample size, a positive integer.
t	a vector with values in $[-1, 1]$ .
scaled	whether to scale the angular function by the von Mises–Fisher normalizing constant. Defaults to TRUE.

## Details

`r_g_vMF` implements algorithm VM in Wood (1994). `c_vMF` is vectorized on `p` and `kappa`.

## Value

Depending on the function:

- `d_vMF`: a vector of length `nx` or 1 with the evaluated density at `x`.
- `r_vMF`: a matrix of size `c(n, p)` with the random sample.
- `c_vMF`: the normalizing constant.
- `g_vMF`: a vector of size `length(t)` with the evaluated angular function.
- `r_g_vMF`: a vector of length `n` containing simulated values from the cosines density associated to the angular function.

## Author(s)

Eduardo García-Portugués, Davy Paindaveine, and Thomas Verdebout.

## References

Wood, A. T. A. (1994) Simulation of the von Mises Fisher distribution. *Commun. Stat. Simulat.*, 23(1):157–164. doi: [10.1080/03610919408813161](https://doi.org/10.1080/03610919408813161)

## See Also

[tangent-vMF](#).

## Examples

```
# Simulation and density evaluation for p = 2
mu <- c(0, 1)
kappa <- 2
n <- 1e3
x <- r_vMF(n = n, mu = mu, kappa = kappa)
col <- viridisLite::viridis(n)
r <- runif(n, 0.95, 1.05) # Radius perturbation to improve visualization
plot(r * x, pch = 16, col = col[rank(d_vMF(x = x, mu = mu, kappa = kappa))])

# Simulation and density evaluation for p = 3
mu <- c(0, 0, 1)
kappa <- 2
x <- r_vMF(n = n, mu = mu, kappa = kappa)
if (requireNamespace("rgl")) {
  rgl::plot3d(x, col = col[rank(d_vMF(x = x, mu = mu, kappa = kappa))],
             size = 5)
}

# Cosines density
g_tilde <- function(t, p, kappa) {
  exp(w_p(p = p - 1, log = TRUE) +
```

```
      g_vMF(t = t, p = p, kappa = kappa, scaled = TRUE, log = TRUE) +
      ((p - 3) / 2) * log(1 - t^2))
}

# Simulated data from the cosines density
n <- 1e3
p <- 3
kappa <- 2
hist(r_g_vMF(n = n, p = p, kappa = kappa), breaks = seq(-1, 1, l = 20),
     probability = TRUE, main = "Simulated data from g_vMF", xlab = "t")
t <- seq(-1, 1, by = 0.01)
lines(t, g_tilde(t = t, p = p, kappa = kappa))

# Cosine density as a function of the dimension
M <- 100
col <- viridisLite::viridis(M)
plot(t, g_tilde(t = t, p = 2, kappa = kappa), col = col[2], type = "l",
     ylab = "Density")
for (p in 3:M) {
  lines(t, g_tilde(t = t, p = p, kappa = kappa), col = col[p])
}
```

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