Definitions of ψ-Functions Available in Robustbase

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Contents

Preamble

Unless otherwise stated, the following definitions of functions are given by Maronna et al. (2006, p. 31), however our definitions differ sometimes slightly from theirs, as we prefer a different way of *standardizing* the functions. To avoid confusion, we first define ψ- and ρ-functions.

Definition 1 *A* ψ-function *is a piecewise continuous function* $\psi : \mathbb{R} \to \mathbb{R}$ *such that*

1. ψ *is odd, i.e.,* $\psi(-x) = -\psi(x)\ \forall x$,
2. $\psi(x) \geq 0$ *for* $x \geq 0$*, and* $\psi(x) > 0$ *for* $0 < x < x_r := \sup\{\tilde{x} : \psi(\tilde{x}) > 0\}$ $\quad$ *(*$x_r > 0$*, possibly* $x_r = \infty$*).*

3* *Its slope is* 1 *at* 0*, i.e.,* $\psi'(0) = 1$.

Note that '3' is not strictly required mathematically, but we use it for standardization in those cases where* ψ *is continuous at 0. Then, it also follows (from 1.) that* $\psi(0) = 0$*, and we require* $\psi(0) = 0$ *also for the case where* ψ *is discontinuous in 0, as it is, e.g., for the M-estimator defining the median.*

Definition 2 *A* ρ-function *can be represented by the following integral of a* ψ*-function,*

$$\rho(x) = \int_0^x \psi(u)du\ , \tag{1}$$

which entails that $\rho(0) = 0$ *and* ρ *is an even function.*

A ψ-function is called *redescending* if $\psi(x) = 0$ for all $x \geq x_r$ for $x_r < \infty$, and x_r is often called *rejection point*. Corresponding to a redescending ψ-function, we define the function $\tilde{\rho}$, a version of ρ standardized such as to attain maximum value one. Formally,

$$\tilde{\rho}(x) = \rho(x)/\rho(\infty). \tag{2}$$

Note that $\rho(\infty) = \rho(x_r) \equiv \rho(x)\ \forall |x| >= x_r$. $\tilde{\rho}$ is a ρ-function as defined in Maronna et al. (2006) and has been called χ function in other contexts. For example, in package `robustbase`, `Mchi(x, *)` computes $\tilde{\rho}(x)$, whereas `Mpsi(x, *, deriv=-1)` ("(-1)-st derivative" is the primitive or antiderivative)) computes $\rho(x)$, both according to the above definitions.

Note: An alternative slightly more general definition of *redescending* would only require $\rho(\infty) := \lim_{x\to\infty} \rho(x)$ to be finite. E.g., `"Welsh"` does *not* have a finite rejection point, but *does* have bounded ρ, and hence well defined $\rho(\infty)$, and we *can* use it in `lmrob()`.[1]

Weakly redescending ψ functions. Note that the above definition does require a finite rejection point x_r. Consequently, e.g., the score function $s(x) = -f'(x)/f(x)$ for the Cauchy ($= t_1$) distribution, which is $s(x) = 2x/(1 + x^2)$ and hence non-monotone and "re descends" to 0 for $x \to \pm\infty$, and $\psi_C(x) := s(x)/2$ also fulfills ${\psi_C}'(0) = 1$, but it has $x_r = \infty$ and hence $\psi_C()$ is *not* a redescending ψ-function in our sense. As they appear e.g. in the MLE for t_ν, we call ψ-functions fulfulling $\lim_{x\to\infty} \psi(x) = 0$ *weakly redescending*. Note that they'd naturally fall into two sub categories, namely the one with a *finite* ρ-limit, i.e. $\rho(\infty) := \lim_{x\to\infty} \rho(x)$, and those, as e.g., the t_ν score functions above, for which $\rho(x)$ is unbounded even though $\rho' = \psi$ tends to zero.

1 Monotone ψ-Functions

Montone ψ-functions lead to convex ρ-functions such that the corresponding M-estimators are defined uniquely.

Historically, the "Huber function" has been the first ψ-function, proposed by Peter Huber in Huber (1964).

[1] E-mail Oct. 18, 2014 to Manuel and Werner, proposing to change the definition of "redescending".

1.1 Huber

The family of Huber functions is defined as,

$$\rho_k(x) = \begin{cases} \frac{1}{2}x^2 & \text{if } |x| \le k \\ k(|x| - \frac{k}{2}) & \text{if } |x| > k \end{cases},$$
$$\psi_k(x) = \begin{cases} x & \text{if } |x| \le k \\ k \ \operatorname{sign}(x) & \text{if } |x| > k \end{cases}.$$

The constant k for 95% efficiency of the regression estimator is 1.345.

```
> plot(huberPsi, x., ylim=c(-1.4, 5), leg.loc="topright", main=FALSE)
```

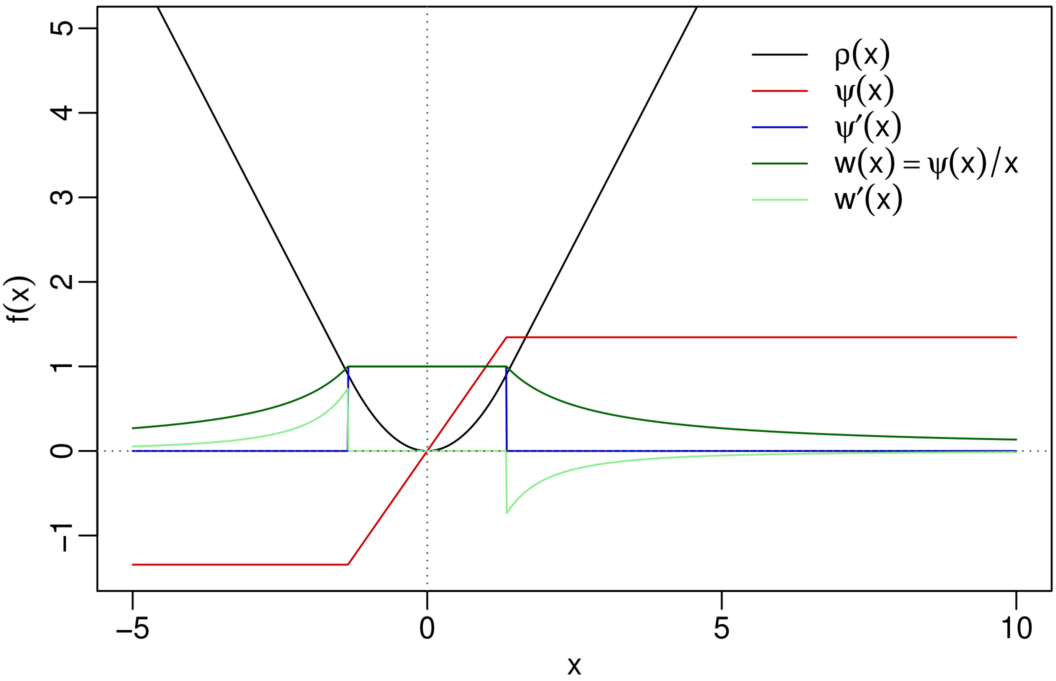

Figure 1: Huber family of functions using tuning parameter $k = 1.345$.

2 Redescenders

For the MM-estimators and their generalizations available via `lmrob()` (and for some methods of `nlrob()`), the ψ-functions are all redescending, i.e., with finite "rejection point" $x_r = \sup\{t; \psi(t) > 0\} < \infty$. From `lmrob`, the psi functions are available via `lmrob.control`, or more directly, `.Mpsi.tuning.defaults`,

```
> names(.Mpsi.tuning.defaults)
```

```
[1] "huber"     "bisquare" "welsh"     "ggw"       "lqq"
[6] "optimal"   "hampel"
```

and their ψ, ρ, ψ', and weight function $w(x) := \psi(x)/x$, are all computed efficiently via C code, and are defined and visualized in the following subsections.

2.1 Bisquare

Tukey's bisquare (aka "biweight") family of functions is defined as,

$$\tilde{\rho}_k(x) = \begin{cases} 1 - \left(1 - (x/k)^2\right)^3 & \text{if } |x| \le k \\ 1 & \text{if } |x| > k \end{cases},$$

with derivative $\tilde{\rho}'_k(x) = 6\psi_k(x)/k^2$ where,

$$\psi_k(x) = x \left(1 - \left(\frac{x}{k}\right)^2\right)^2 \cdot I_{\{|x| \le k\}}\ .$$

The constant k for 95% efficiency of the regression estimator is 4.685 and the constant for a breakdown point of 0.5 of the S-estimator is 1.548. Note that the *exact* default tuning constants for M- and MM- estimation in `robustbase` are available via `.Mpsi.tuning.default()` and `.Mchi.tuning.default()`, respectively, e.g., here,

```
> print(c(k.M = .Mpsi.tuning.default("bisquare"),
+         k.S = .Mchi.tuning.default("bisquare")), digits = 10)
```

```
     k.M      k.S
4.685061 1.547640
```

and that the `p.psiFun(.)` utility is available via

```
> source(system.file("xtraR/plot-psiFun.R", package = "robustbase", mustWork=TRUE))
```

```
> p.psiFun(x., "biweight", par = 4.685)
```

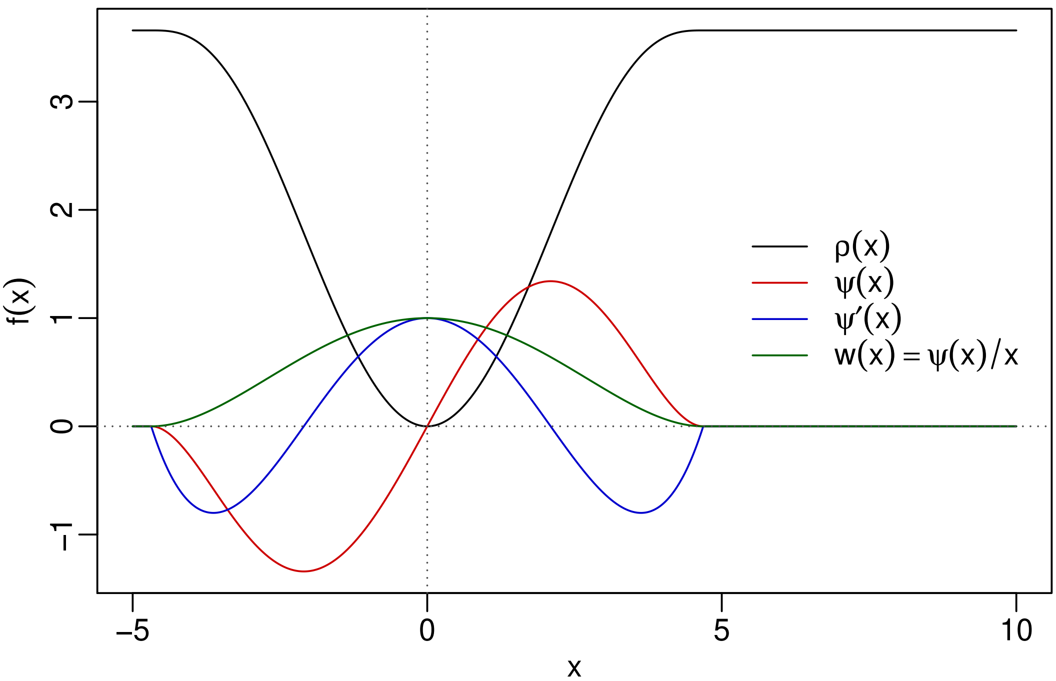

Figure 2: Bisquare family functions using tuning parameter $k = 4.685$.

2.2 Hampel

The Hampel family of functions (Hampel et al., 1986) is defined as,

$$\tilde{\rho}_{a,b,r}(x) = \begin{cases} \frac{1}{2}x^2/C & |x| \le a \\ \left(\frac{1}{2}a^2 + a(|x|-a)\right)/C & a < |x| \le b \\ \frac{a}{2}\left(2b - a + (|x|-b)\left(1 + \frac{r-|x|}{r-b}\right)\right)/C & b < |x| \le r \\ 1 & r < |x| \end{cases},$$

$$\psi_{a,b,r}(x) = \begin{cases} x & |x| \le a \\ a \ \operatorname{sign}(x) & a < |x| \le b \\ a \ \operatorname{sign}(x)\frac{r-|x|}{r-b} & b < |x| \le r \\ 0 & r < |x| \end{cases},$$

where $C := \rho(\infty) = \rho(r) = \frac{a}{2}\left(2b - a + (r-b)\right) = \frac{a}{2}(b - a + r)$.

As per our standardization, ψ has slope 1 in the center. The slope of the redescending part ($x \in [b, r]$) is $-a/(r-b)$. If it is set to $-\frac{1}{2}$, as recommended sometimes, one has

$$r = 2a + b\ .$$

Here however, we restrict ourselves to $a = 1.5k$, $b = 3.5k$, and $r = 8k$, hence a redescending slope of $-\frac{1}{3}$, and vary k to get the desired efficiency or breakdown point.

The constant k for 95% efficiency of the regression estimator is 0.902 (0.9016085, to be exact) and the one for a breakdown point of 0.5 of the S-estimator is 0.212 (i.e., 0.2119163).

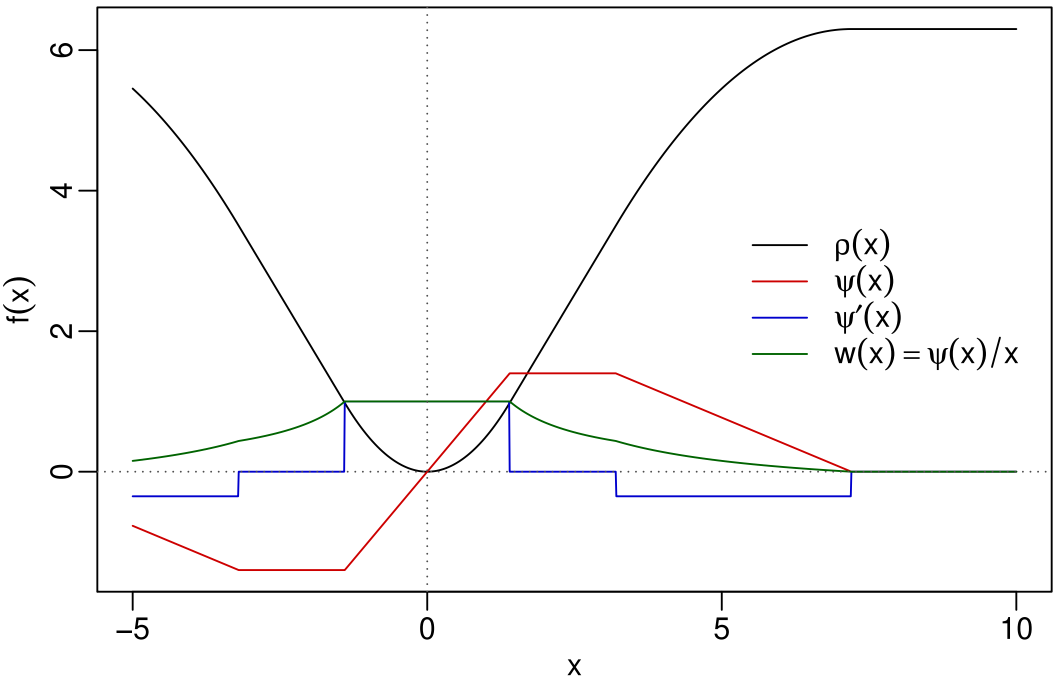

Figure 3: Hampel family of functions using tuning parameters $0.902 \cdot (1.5, 3.5, 8)$.

2.3 GGW

The Generalized Gauss-Weight function, or *ggw* for short, is a generalization of the Welsh ψ-function (subsection 2.6). In Koller and Stahel (2011) it is defined as,

$$\psi_{a,b,c}(x) = \begin{cases} x & |x| \le c \\ \exp\left(-\frac{1}{2}\frac{(|x|-c)^b}{a}\right) x & |x| > c \end{cases}.$$

Our constants, fixing $b = 1.5$, and minimial slope at $-\frac{1}{2}$, for 95% efficiency of the regression estimator are $a = 1.387$, $b = 1.5$ and $c = 1.063$, and those for a breakdown point of 0.5 of the S-estimator are $a = 0.204$, $b = 1.5$ and $c = 0.296$:

```
> cT <- rbind(cc1 = .psi.ggw.findc(ms = -0.5, b = 1.5, eff = 0.95        ),
+             cc2 = .psi.ggw.findc(ms = -0.5, b = 1.5,           bp = 0.50)); cT
```

```
    [,1]      [,2] [,3]      [,4]      [,5]
cc1    0 1.3863620  1.5 1.0628199 4.7773893
cc2    0 0.2036739  1.5 0.2959131 0.3703396
```

Note that above, `cc*[1]`$= 0$, `cc*[5]`$= \rho(\infty)$, and `cc*[2:4]`$= (a, b, c)$. To get this from (a, b, c), you could use

```
> ipsi.ggw <- .psi2ipsi("GGW") # = 5
> ccc <- c(0, cT[1, 2:4], 1)
> integrate(.Mpsi, 0, Inf, ccc=ccc, ipsi=ipsi.ggw)$value # = rho(Inf)
```

```
[1] 4.777389
```

```
> p.psiFun(x., "GGW", par = c(-.5, 1, .95, NA))
```

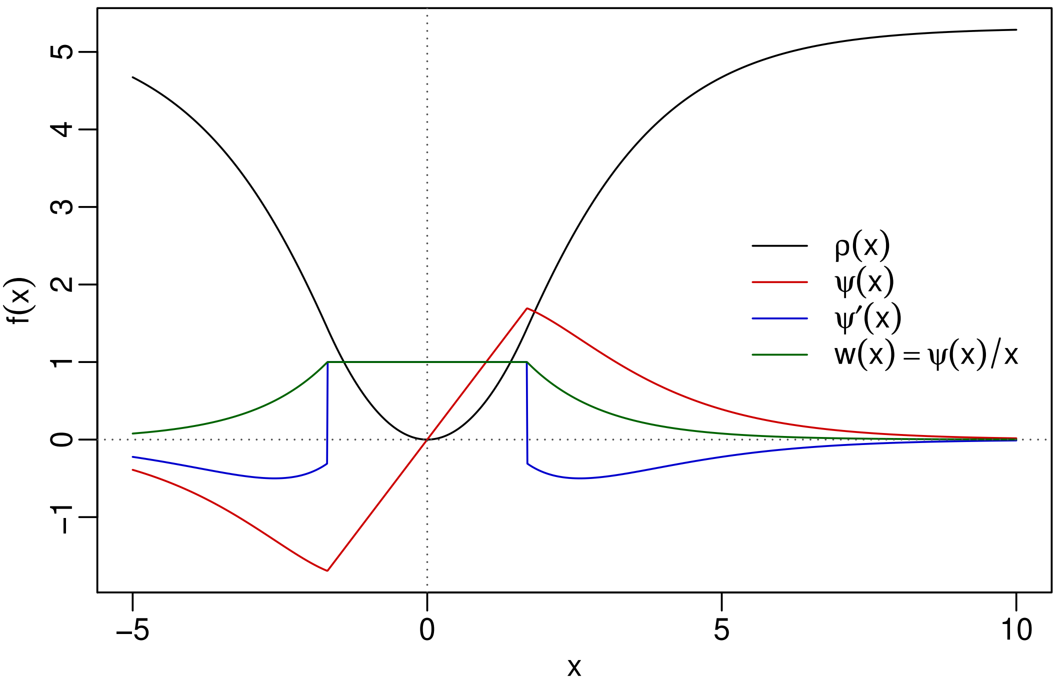

Figure 4: GGW family of functions using tuning parameters $a = 1.387$, $b = 1.5$ and $c = 1.063$.

2.4 LQQ

The "linear quadratic quadratic" ψ-function, or *lqq* for short, was proposed by Koller and Stahel (2011). It is defined as,

$$\psi_{b,c,s}(x) = \begin{cases} x & |x| \le c \\ \mathrm{sign}(x)\left(|x| - \frac{s}{2b}\left(|x|-c\right)^2\right) & c < |x| \le b + c \\ \mathrm{sign}(x)\left(c + b - \frac{bs}{2} + \frac{s-1}{a}\left(\frac{1}{2}\tilde{x}^2 - a\tilde{x}\right)\right) & b + c < |x| \le a + b + c \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\tilde{x} := |x| - b - c \quad \text{and} \quad a := (2c + 2b - bs)/(s-1). \tag{3}$$

The parameter c determines the width of the central identity part. The sharpness of the bend is adjusted by b while the maximal rate of descent is controlled by s ($s = 1 - \min_x \psi'(x) > 1$). From (3), the length a of the final descent to 0 is a function of b, c and s.

```
> cT <- rbind(cc1 = .psi.lqq.findc(ms= -0.5, b.c = 1.5, eff=0.95, bp=NA ),
+             cc2 = .psi.lqq.findc(ms= -0.5, b.c = 1.5, eff=NA , bp=0.50))
> colnames(cT) <- c("b", "c", "s"); cT

            b         c   s
cc1 1.4734061 0.9822707 1.5
cc2 0.4015457 0.2676971 1.5
```

If the minimal slope is set to $-\frac{1}{2}$, i.e., $s = 1.5$, and $b/c = 3/2 = 1.5$, the constants for 95% efficiency of the regression estimator are $b = 1.473$, $c = 0.982$ and $s = 1.5$, and those for a breakdown point of 0.5 of the S-estimator are $b = 0.402$, $c = 0.268$ and $s = 1.5$.

```
> p.psiFun(x., "LQQ", par = c(-.5,1.5,.95,NA))
```

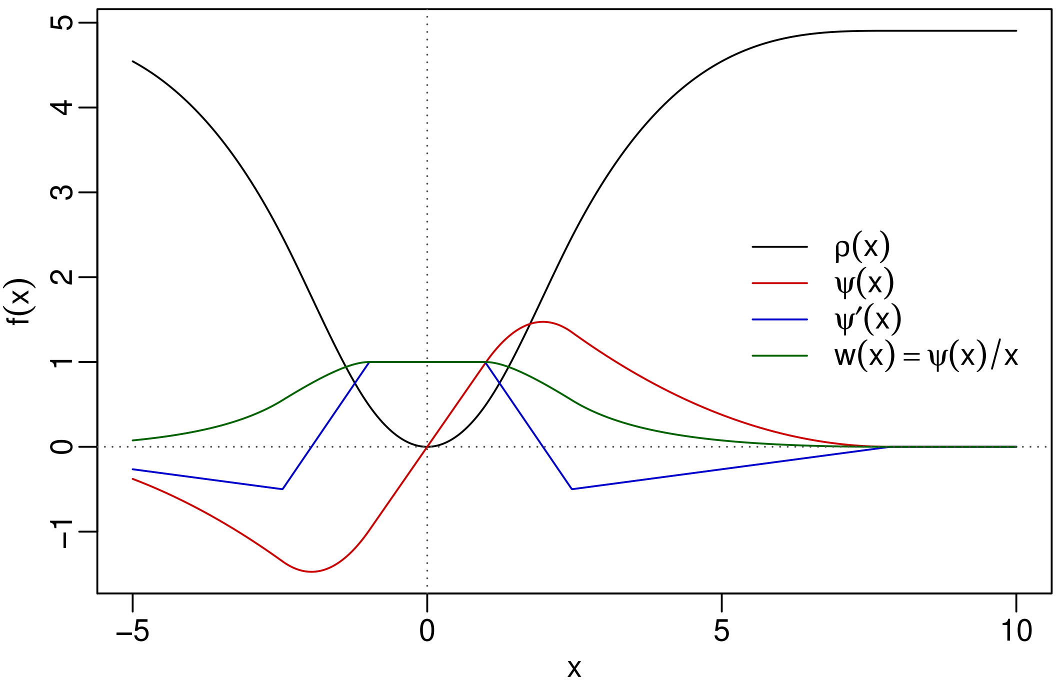

Figure 5: LQQ family of functions using tuning parameters $b = 1.473$, $c = 0.982$ and $s = 1.5$.

2.5 Optimal

The optimal ψ function as given by Maronna et al. (2006, Section 5.9.1),

$$\psi_c(x) = \mathrm{sign}(x) \left(-\frac{\varphi'(|x|) + c}{\varphi(|x|)} \right)_+ ,$$

where φ is the standard normal density, c is a constant and $t_+ := \max(t, 0)$ denotes the positive part of t.

Note that the `robustbase` implementation uses rational approximations originating from the `robust` package's implementation. That approximation also avoids an anomaly for small x and has a very different meaning of c.

The constant for 95% efficiency of the regression estimator is 1.060 and the constant for a breakdown point of 0.5 of the S-estimator is 0.405.

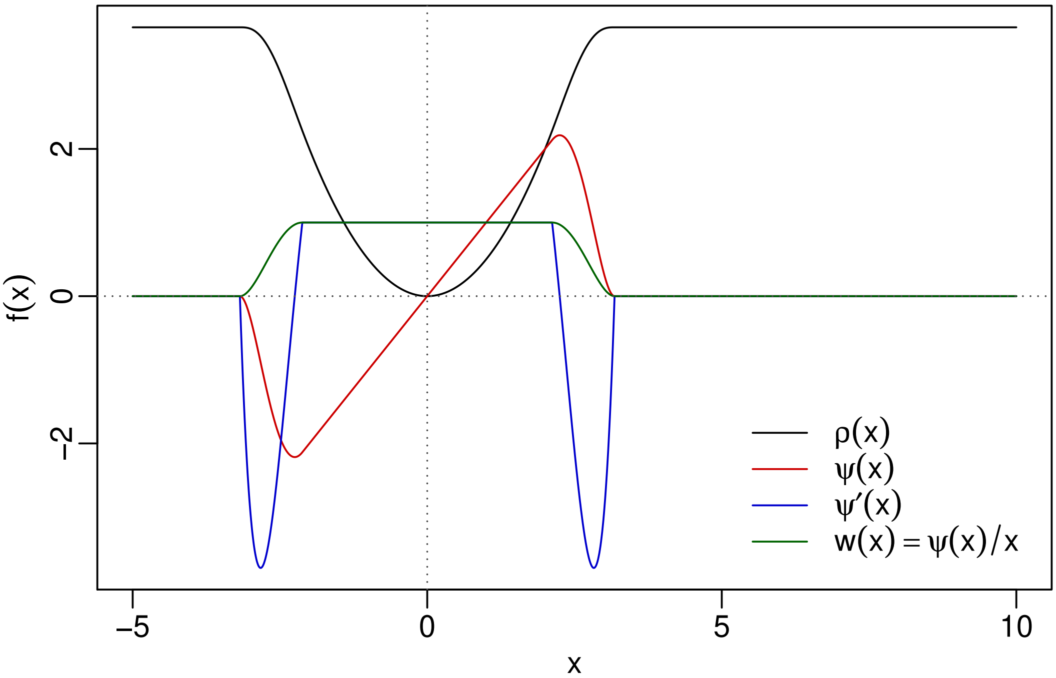

Figure 6: 'Optimal' family of functions using tuning parameter $c = 1.06$.

2.6 Welsh

The Welsh ψ function is defined as,

$$\tilde{\rho}_k(x) = 1 - \exp\big(-\left(x/k\right)^2/2\big)$$
$$\psi_k(x) = k^2\tilde{\rho}'_k(x) = x\exp\big(-\left(x/k\right)^2/2\big)$$
$$\psi'_k(x) = \big(1-\left(x/k\right)^2\big)\exp\big(-\left(x/k\right)^2/2\big)$$

The constant k for 95% efficiency of the regression estimator is 2.11 and the constant for a breakdown point of 0.5 of the S-estimator is 0.577.

Note that GGW (subsection 2.3) is a 3-parameter generalization of Welsh, matching for $b=2$, $c=0$, and $a=k^2$ (see R code there):

```
> ccc <- c(0, a = 2.11^2, b = 2, c = 0, 1)
> (ccc[5] <- integrate(.Mpsi, 0, Inf, ccc=ccc, ipsi = 5)$value) # = rho(Inf)

[1] 4.4521

> stopifnot(all.equal(Mpsi(x., ccc,  "GGW"),   ## psi[ GGW ](x; a=k^2, b=2, c=0) ==
+                     Mpsi(x., 2.11, "Welsh")))## psi[Welsh](x; k)
```

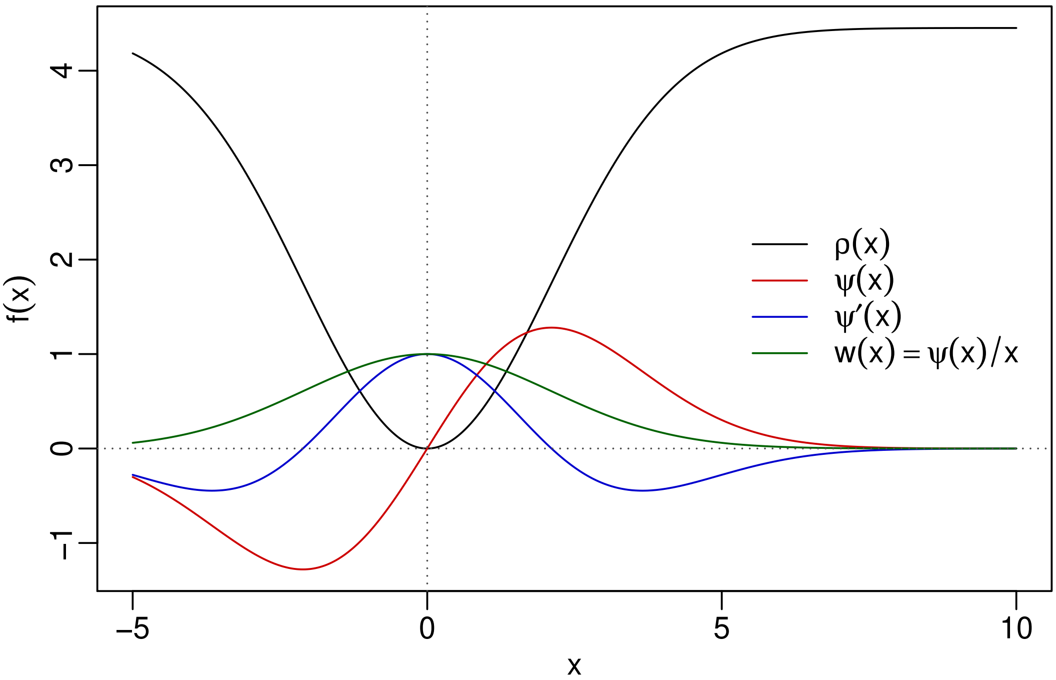

Figure 7: Welsh family of functions using tuning parameter $k = 2.11$.

References

Hampel, F., E. Ronchetti, P. Rousseeuw, and W. Stahel (1986). *Robust Statistics: The Approach Based on Influence Functions*. N.Y.: Wiley.

Huber, P. J. (1964). Robust estimation of a location parameter. *Ann. Math. Statist. 35*, 73–101.

Koller, M. and W. A. Stahel (2011). Sharpening wald-type inference in robust regression for small samples. *Computational Statistics & Data Analysis 55*(8), 2504–2515.

Maronna, R. A., R. D. Martin, and V. J. Yohai (2006). *Robust Statistics, Theory and Methods*. Wiley Series in Probility and Statistics. John Wiley & Sons, Ltd.